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The coherency index

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Abstract

A *coherent decomposition* of a metric is, in general, a natural decomposition of the metric into a sum of simpler metrics. A metric that has only a trivial coherent decomposition is called *prime*. In this paper we give a formula for an index, called the *coherency index*, which allows us to prove that there exist only finitely many prime metrics (up to multiplication by a positive scalar) on a finite set. Moreover, the formula that we give for the coherency index also provides us with a computational tool by which one can compute coherent decompositions of metrics into primes.

1. Introduction

One of the central problems in the subject of discrete metric spaces is to understand the *structure* of the metric cone on a finite set X, which we denote by $M(X)$. One approach to this problem is to analyze the *extremal rays* of $M(X)$ and much work has been done in this direction (see for example [1,2,4,11]). In this paper we take a different approach, using the concept of *coherent decompositions*, an idea that was originally suggested by Bandelt and Dress in [3].

Generally speaking, a coherent decomposition of a metric is a natural decomposition of the metric into a sum a simpler metrics. We are mainly interested in coherent decompositions of metrics into sums of *prime metrics*, that is metrics that cannot be coherently decomposed in a non-trivial way. In [3] it was conjectured that coherently

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decomposing metrics into primes should enable one to express the metric cone as the union of a set of simplicial subcones. This was based, in part, on a general analysis of five-point metrics, originally given in [5]. However, in general this breaks down even on a set of cardinality six. For example, in [13] a six-point metric is given which has two distinct coherent decompositions into linearly independent primes. This 'non-uniqueness of factorization' is somewhat analogous to factorization in the ring $\mathbb{Z}[\sqrt{-5}]$, which is not a unique factorization domain (for example, in this ring $21 = 3 \cdot 7 = (1 + 2\sqrt{-5}) \cdot (1 - 2\sqrt{-5})$), but which still admits finitely many factorizations of each of its elements into primes.

However, as we shall see, all is not lost. One of the main results in this paper is that there exist only finitely many prime metrics on a finite set (where, of course, we consider two metrics which differ by a product of a positive scalar as being equivalent). As an immediate consequence of this result it can be seen that, up to some notion of equivalence, any metric on a finite set has only finitely many coherent decompositions into primes.

To prove these facts we define and give a formula for an index, called the *coherency index*, an index whose existence is shown in [7]. This index also enables one to compute the coherent decompositions of a metric on a finite set into primes, once the set of all primes on that set are known. Hence, in theory at least, it is possible to find all of the coherent decompositions of any given metric into primes. As we show, the coherency index is a direct generalization of the *isolation index*, an index that was defined in [3] and which allows one to decompose metrics into linear combinations of *split metrics*. One of the main advantages of the coherency index (and the isolation index) is its computability. In a follow up paper we will use this fact to classify all prime metrics on six points, and to analyse coherent decompositions of six-point metrics. Further, in this paper, we give a one-to-one correspondence between the so-called maximal coherent decompositions, which we introduce in this paper, and certain tight spans.

2. Preliminaries

2.1. Metrics

Let d be a *metric* defined on a set X, that is, a map $d: X \times X \to \mathbb{R}$ such that $d(x,x) = 0$, $d(x,y) = d(y,x)$ and $d(x,z) \leqslant d(x,y) + d(z,y)$ for all $x,y,z \in X$. The metric d is called *proper* if the condition $d(x,y) = 0$ implies that $x = y$, for all $x,y \in X$. When X is a finite set, as will always be the case in this paper, the pair (X,d) is called a *finite metric space*. The metrics d, d' on X are of *same type* if there exist a $\lambda > 0$ and a permutation τ of X such that

$$d(x,y) = \lambda d'(\tau(x), \tau(y))$$

for all $x, y \in X$.

A metric d defines an equivalence relation $\sim_d$ on the set X, by setting $x \sim_d y$ if and only if $d(x,y)=0$ for all $x,y \in X$. We define a proper metric on the set of equivalence classes $X/\sim_d$ by setting $d(A,B)=d(x,y)$, where $A,B \in X/\sim_d$, and $x \in A$, $y \in B$. This metric is clearly well defined, and we call it the *zero contraction* of d (see [1]). If Y is a subset of X, then a metric d on Y is called an *extension* of d' if $d' = d|_{Y \times Y}$.

2.2. Decompositions

For a given finite metric space (X,d), a finite set of metrics $\{d_1,\ldots,d_n\}$ on X is called a *decomposition of d* if

$$d(x,y)=d_1(x,y)+\cdots+d_n(x,y),$$

for all $x,y \in X$. We often denote this decomposition of d by $d=d_1+\cdots+d_n$.

Let D be a finite set of metrics on X. A metric d on X is of *type D* if $d=\sum_{d' \in D} \alpha_{d'} \cdot d'$, where $\alpha_{d'} \geqslant 0$. If $\alpha_{d'}>0$ for all $d' \in D$, then we say that d is of *generic type D*. The metric d is of *isomorphism type D* if there is a metric d' of type D that is isomorphic to d.

2.3. Graphs

Let $G=(V,E)$ be a graph with vertex set V and edge set E contained in V^2. A *path* in G is a subset $\{v_0,\ldots,v_m\}$ of distinct vertices in V such that $\{v_i,v_{i+1}\}$ is an edge of G for all $0 \leqslant i \leqslant m-1$. Such a path has *length m*. Each graph G induces a metric on the set V which assigns the length of a shortest path or geodesic in G to any pair of vertices in V.

3. Coherent decompositions and prime metrics

In this section we recall the concept of a coherent decomposition, and related topics. This concept was originally introduced in [3], and was motivated in part by the *tight span* of a metric space, an object which was initially discovered in [12] and subsequently rediscovered and analyzed in [5]. The study and use of this object is central in *T-theory*, a new field that is concerned with the mathematics of similarity [9], and, since it is of vital importance in the definition of the coherency index, we now recall some of its properties.

3.1. The tight span

Given a finite metric space (X,d) we define its *associated polytope*, $P(d)=P(X,d)$, which is contained in $\mathbb{R}^X$, by

$$P(X,d):=\{f:X \to \mathbb{R} \mid f(x)+f(y) \geqslant d(x,y), \text{ for all } x,y \in X\}.$$

The union of the compact faces of this polytope, which we denote by $T(d)=T(X,d)$, is called the *tight span or T-construction of* (X,d). Thus, in general, $T(d)$ should be regarded as being a non-convex, bounded polytope contained in $\mathbb{R}^{|X|}$. We let $VT(d)$ denote the vertices of this polytope.

The tight span has many nice properties (see for example [5,6,9,10]), however we list here only those which are of interest to us in this paper.

Property 1. The tight span $T(d)$ is also equal to the following set of functions [5, Theorem 3]:

$$\left\{ f : X \to \mathbb{R} \mid f(x) = \sup_{y\in X} (d(x,y) - f(y)) \text{ for all } x \in X \right\}.$$

Moreover, it is also proven in [5] that, if one defines the map

$$d_\infty : T(d) \times T(d) \to \mathbb{R},$$

$$d_\infty : (f,g) \mapsto \sup_{x\in X} |f(x) - g(x)|,$$

then $(T(d), d_\infty)$ is a metric space, and that there exists a canonical isometric embedding of (X,d) into $(T(d), d_\infty)$ given by

$$t_d : X \to T(d),$$

$$t_d : x \mapsto h_x,$$

where the function $h_x \in T(d)$ is defined by setting

$$h_x(y) := d(x,y),$$

for all $y \in X$.

Property 2. Associate a graph $K(f)$ to each element $f \in P(d)$ as follows. The vertex set of $K(f)$ is equal to X, and its edge set consists of those $\{x,y\} \in X^2$ such that $f(x)+f(y)=d(x,y)$. We let $EK(f)$ denote the set of edges of $K(f)$. In [6, Lemma 2] it is proven that the number of bipartite connected components of $K(f)$ is equal to the dimension of the minimal face of $T(d)$ in which f is contained. Furthermore, it follows from Property 1 that $f \in T(d)$ if and only if for all $x \in X$ there exists a $y \in X$ such that $\{x,y\}$ is an edge of $K(f)$.

Property 3. Let Y be a subset of X, and $d_1 := d|_{Y\times Y}$. Then for all $f \in T(d_1)$ there exists some $g \in T(d)$ such that $g|_Y = f$ [5, Theorem 3(vi)]. Moreover, it also follows that if $f \in VT(d_1)$, then there exists some $g \in VT(d)$ such that $g|_Y = f$.

Property 4. The tight span $T(d)$ is equal to the set of minimal elements of $P(d)$ with respect to the ordering $\leqslant$, defined by setting $f \leqslant g$ if and only if $g(x) \leqslant f(x)$ for all $x \in X$ [5, Theorem 3].

Property 5. Let $VT(d)$ denote the set of vertices of $T(d)$. Since $T(d)$ consists of the compact faces of $P(d)$, the set $VT(d)$ is equal to the set of vertices of $P(d)$.

Property 6. For a function $f \in P(d)$ the set $\{g \in P(d) \mid K(f)$ is a subgraph of $K(g)\}$ is a face of $P(d)$ (see [6]). It is the smallest face of $P(d)$ in which f lies.

3.2. Coherent decompositions

Let $\{d_1, \dots, d_n\}$ be a finite set of metrics on X and define $d := d_1 + \cdots + d_n$. Then we clearly have the inclusion

$$P(d_1) + \cdots + P(d_n) \subseteq P(d),$$

where

$$P(d_1) + \cdots + P(d_n) := \{f_1 + \cdots + f_n \mid f_i \in P(d_i), \text{ for } 1 \leqslant i \leqslant n\}.$$

If equality holds in the above inclusion, then we call $\{d_1, \dots, d_n\}$ a *coherent decomposition of* d [3]. A coherent decomposition $\{d', d''\}$ of d is called *trivial* if $d' = \alpha d$ for some non-negative number α. A metric is called *prime* if it has no non-trivial coherent decomposition $\{d', d''\}$. Given a finite metric space (X, d), a non-zero metric d' on X is called a *coherent component* of d if there exists a $\lambda > 0$ and a metric d'' such that $\lambda d' + d'' = d$ is a coherent decomposition of d.

3.3. Criteria for checking coherency

In the next two lemmas we give criteria for checking whether or not a decomposition is coherent.

Lemma 3.1 (Bandelt and Dress [3, p. 89]). *The decomposition $d = d_1 + \cdots + d_n$ is coherent if and only if $T(d)$ is a subset of $T(d_1) + \cdots + T(d_n)$.*

Lemma 3.2 (Koolen and Moulton [13, Lemma 1.2]). *The decomposition $d = d_1 + \cdots + d_n$ is coherent if and only if every vertex f of $T(d)$ can be written as the sum $f = f_1 + \cdots + f_n$, where $f_i \in T(d_i)$, $1 \leqslant i \leqslant n$.*

Corollary 3.3. *Suppose that $\{d_1, \dots, d_n\}$ is a coherent decomposition of d and that $f \in VT(d)$. Then $f = f_1 + \cdots + f_n$, where $f_i \in VT(d_i)$ for all $1 \leqslant i \leqslant n$.*

Proof. Since $f \in VT(d)$, by Lemma 3.2 $f = f_1 + \cdots + f_n$, where $f_i \in T(d_i)$. Moreover, it follows that if $f(x) + f(y) = d(x, y)$, then $f_i(x) + f_i(y) = d_i(x, y)$ for all $1 \leqslant i \leqslant n$. $\square$

3.4. A fundamental property of coherent decompositions

Proposition 3.4. *If $\{d_1,\dots,d_n\}$ is a coherent decomposition of d, then for $0\leqslant\alpha\leqslant 1$, the decomposition of $d' := \alpha d_1 + d_2 + \cdots + d_n$ is also coherent.*

Proof. Let $d := d_1 + \cdots + d_n$, $d' := \alpha d_1 + d_2 + \cdots + d_n$, and f' be an element of $T(d')$. We will show that

$$f' = \alpha f_1 + \sum_{i=2}^{m} f_i,$$

where $f_i \in P(d_i)$, $1\leqslant i\leqslant m$, which proves the proposition as a consequence of Lemma 3.1.

Let $g_1,\dots,g_m$ denote the vertices of $T(d_1)$. Then

$$(1-\alpha)g_i + f'$$

is an element of $P(d)$, for all $1\leqslant i\leqslant m$. Since $\{d_1,\dots,d_n\}$ is a coherent decomposition of d, by Lemma 3.1, we can rewrite this sum as

$$(1-\alpha)g_i + f' = \sum_{j=1}^{n} f_{ij},$$

where $f_{ij} \in P(d_j)$, $1\leqslant j\leqslant n$. Without loss of generality, we may assume that $f_{i1} \in T_1(d_1)$ for $1\leqslant i\leqslant m$. Moreover, since $T(d_1)$ is a union of convex polytopes,

$$f_{i1} = \sum_{j=1}^{m} a_{ij} g_j,$$

where $a_{ij}\geqslant 0$, and $\sum_{j=1}^{m} a_{ij} = 1$.

By Perron–Frobenious Theory (see [15, Theorem 0.16], for example) there exists a non-negative left eigenvector $(v_1,\dots,v_m)$ of the matrix (a_{ij}), with eigenvalue 1. Without loss of generality, we can also assume that $\sum_{i=1}^{m} v_i = 1$. Hence,

$$\begin{aligned}(1-\alpha)\sum_{i=1}^{m} v_i g_i + f' &= \sum_{i=1}^{m} v_i((1-\alpha)g_i + f') \\ &= \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{m} a_{ij} g_j + \sum_{j=2}^{m} f_{ij}\right) \\ &= \sum_{j=1}^{m}\sum_{i=1}^{m} v_i a_{ij} g_j + \sum_{j=2}^{m}\sum_{i=1}^{m} v_i f_{ij} \\ &= \sum_{j=1}^{m} v_j g_j + \sum_{j=2}^{m}\sum_{i=1}^{m} v_i f_{ij},\end{aligned}$$

and thus

$$f' = \alpha \sum_{i=1}^{m} v_i g_i + \sum_{j=2}^{m}\sum_{i=1}^{m} v_i f_{ij}.$$

Set $f_1 := \sum_{i=1}^m v_i g_i$ and $f_j := \sum_{i=1}^m v_i f_{ij}$, where $2 \leqslant j \leqslant n$. Then, since

$$\sum_{i=1}^{m} v_i = 1,$$

we see that $f_j \in P(d_j)$ for all $1 \leqslant j \leqslant n$, which implies that $f_j \in T(d_j)$ for all $1 \leqslant j \leqslant n$, since $f' \in T(d')$, which completes the proof. □

Corollary 3.5. *If $\{d_1, \ldots, d_n\}$ is a coherent decomposition of d, then the decomposition $d' := \alpha_1 d_1 + \cdots + \alpha_n d_n$ is also coherent, where $\alpha_i \geqslant 0$ for all $1 \leqslant i \leqslant n$.*

A finite set of metrics D on a set X is called *coherent* if the decomposition

$$d := \sum_{d' \in D} d'$$

is coherent. Hence, we can restate Corollary 3.5 as follows: if D is a coherent set of metrics then any set of metrics of same type as D is also coherent. This corollary was also shown by Zeng in his Ph.D. thesis [17]. A coherent set $\{d_1, \ldots, d_n\}$ consisting of prime metrics is called *complete* if, when d' is a prime component of $d := d_1 + \cdots + d_n$ then there exist $1 \leqslant i \leqslant n$ and $\alpha > 0$ such that $d' = \alpha d_i$. A complete coherent set $\{d_1, \ldots, d_n\}$ is called *maximal* if for all prime metrics d' the set $\{d_1, \ldots, d_n, d'\}$ is not coherent unless $d' = \alpha d_i$, for some $1 \leqslant i \leqslant n$ and $\alpha > 0$. As we shall see (Corollary 4.15) every metric d has a complete decomposition.

3.5. The split decomposition

One of the most important examples of a coherent decomposition is the *split decomposition*, which was originally introduced by Bandelt and Dress in [3]. Since we will use this decomposition later, we now give a brief introduction to it here. The reader is referred to [3] for more details.

A *split* of a finite set X is a bipartition of X. We denote the set of splits of X by $\mathscr{S}(X)$. For each split $S := \{A, B\}$, $A, B \subseteq X$, we can define the *split metric* associated to S by

$$\delta_S(x, y) := \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in B, \\ 1 & \text{otherwise} \end{cases}$$

(this metric is sometimes also called a *cut metric* — for example, see [11]). The main result of [3] implies that for any metric d there exists a (unique) coherent decomposition

$$d = d_0 + \sum_{S \in \mathscr{S}(X)} \alpha_S^d \cdot \delta_S, \tag{1}$$

where $\alpha_S^d \geqslant 0$, and d_0 cannot be written as the coherent sum of a metric plus a positive multiple of a split metric. This decomposition is called the *split decomposition* of d.

The constant α_S^d is called the *isolation index* of the split $S=\{A,B\}$ with respect to d, and, as is shown in [3], can be computed in the following way. Let Q denote the set of quartets $q=\{a,a';b,b'\}$ contained in X such that $a,a'\in A$ and $b,b'\in B$. If for each quartet $q\in Q$, we define the quantity

$$\alpha_q := \max\{d(a,b)+d(a',b'), d(a,b')+d(a',b), d(a,a')+d(b,b')\} - (d(a,a')+d(b,b')), \tag{2}$$

then

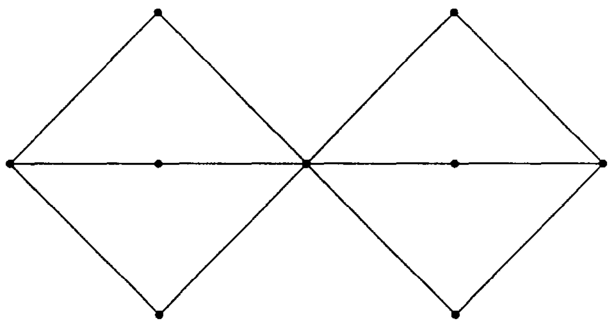
$$\alpha_S^d = \tfrac{1}{2}\min_{q\in Q}\alpha_q.$$

The metric d is called *split-prime* if it has no coherent components that are split metrics. The coherent component d_0 of d defined by Eq. (1) is the *split-prime component* of d. The metric d is split-prime precisely when $d=d_0$. At the other extreme, if the split-prime component of d vanishes, then we call d *totally decomposable.*

3.6. Prime metrics

The concept of a coherent decomposition naturally gives rise to the definition of a *prime* metric: a metric that does not have a non-trivial coherent decomposition. We do not regard the metric on a single point as being prime.

In general, a prime metric is also split-prime (excluding, of course, split metrics) however there exist split-prime metrics which are not prime. For example, in [3, p. 94] it is shown that the following graph metric on nine vertices is split-prime. However,this



metric has a coherent decomposition into two prime $K_{2,3}$ coherent components. This is an example of a more general coherent decomposition called the *block decomposition* (see [8,9] for more details).

Simple examples of prime metrics are furnished by the following lemma, which is an analogue of [1, Theorem 4.2] (stating that a metric is extremal if and only if the zero contraction of it is also extremal).

Lemma 3.6 (Koolen and Moulton [13, Lemma 1.1]). *A metric is prime if and only if any zero contraction of it also is prime.*

Other examples of prime metrics can be constructed in various ways. For example, in [3, Proposition 4], it is shown that the Cartesian product of any three non-trivial connected graphs yields a prime metric. In [14] we will give other examples of primes, including those $K_{n_1,\dots,n_m}$ graphs that are prime, and a classification of the prime metrics on six points.

4. The coherency index

In this section we give a formula for the coherency index of a metric d with respect to a metric d', both of which we assume to be defined on a finite set X. As we shall see, this index is a direct generalization of the isolation index.

4.1. A formula for the coherency index

Suppose that (X,d) is a finite metric space, and that d' is also a metric defined on X. The *coherency index of d with respect to d'*, denoted $\alpha_{d'}^{d}$, is given as follows: For each $f \in T(d)$ and $f' \in T(d')$ define

$$m(f,f') := \min_{\{x,y\} \notin EK(f')} \left\{ \frac{f(x)+f(y)-d(x,y)}{f'(x)+f'(y)-d'(x,y)} \right\},$$

and set

$$\alpha_{d'}^{d} := \min_{f \in VT(d)} \left\{ \max_{f' \in VT(d')} \{m(f,f')\} \right\}.$$

Using this definition we immediately obtain the following theorem which is analogous to [3, Theorem 7].

Theorem 4.1. *Let d,d' be two metrics on X. Then $d=\alpha d'+d_1$ is a coherent decomposition of d if and only if $0 \leqslant \alpha \leqslant \alpha_{d'}^{d}$.*

Proof. Suppose that $d=\alpha d'+d_1$ is coherent. By Lemma 3.2, this is equivalent to the following pair of conditions holding:

(i) For all $f \in VT(d)$, there exists a vertex $g \in VT(d')$ such that $(f-\alpha g) \in VT(d_1)$, and

(ii) (X,d_1) is a metric space.

First we show that (ii) follows from (i). Let $x,y,z \in X$. We want to show that

$$d_1(x,y) \leqslant d_1(x,z)+d_1(y,z).$$

Without loss of generality, we may assume that $\alpha > 0$. Consider the vertex $h_z \in VT(d)$ with $h_z(u) = d(z,u)$ for $u \in X$. Then there exists a vertex $G \in VT(d')$ such that $h_z - \alpha G \in VT(d_1)$. This implies that $G(z) = 0$ and hence $G(u) = d'(z,u)$ for all $u \in X$. Hence

$$d_1(x,y) \leqslant (h_z - \alpha G)(x) + (h_z - \alpha G)(y) = d_1(x,z) + d_1(y,z).$$

Now let $f \in VT(d)$ and $g \in VT(d')$ be such that $(f - \alpha g) \in VT(d_1)$. This implies that for all $x, y \in X$ we have

$$(d - \alpha d')(x,y) \leqslant (f - \alpha g)(x) + (f - \alpha g)(y).$$

But this is equivalent to the inequality

$$\alpha(g(x) + g(y) - d'(x,y)) \leqslant f(x) + f(y) - d(x,y).$$

Hence, if $g(x) + g(y) - d'(x,y) \neq 0$, then

$$\alpha \leqslant \frac{f(x) + f(y) - d(x,y)}{g(x) + g(y) - d'(x,y)}.$$

It now easily follows that (i) is equivalent to $0 \leqslant \alpha \leqslant \alpha_{d'}^{d}$, which completes the proof. □

Remarks. (i) In [7] Dress defines a slightly different coherency index. For two given metrics d and d', he defines the coherency index $\alpha_{d'}^{d}$ as the maximum number α such that $\{d - \alpha d', d'\}$ is a coherent set. His coherency index can take negative values, whereas ours is always non-negative, however when his coherency index is positive it is equal to ours and vice versa. (ii) Consider *distance functions* on a finite set X, i.e. functions $d : X \times X \to \mathbb{R}_{\geqslant 0}$ such that $d(x,x) = 0$ for all $x \in X$ and $d(x,y) = d(y,x)$ for all $x, y \in X$. Then Theorem 4.1 also holds for distance functions. Moreover, if d and d' are metrics and $d = \alpha d' + d''$ is coherent, where d'' is a distance function, then d'' is also a metric. This shows that the coherency index described by Dress in [7] also exists for distance functions.

4.2. Relating the isolation index to the coherency index

As an immediate consequence of Theorem 4.1 and [3, Theorem 7], for any split $S \in \mathscr{S}(X)$ and any metric d on a finite set X, we see that the isolation index and the coherency index coincide, that is

$$\alpha_S^d = \alpha_{\delta_S}^d.$$

It is instructive to see a direct proof of this fact, which we now give.

The proof rests mainly on Property 3 of the tight span, but first we require a little more background on the tight span of a four-point metric space. Following [5] we

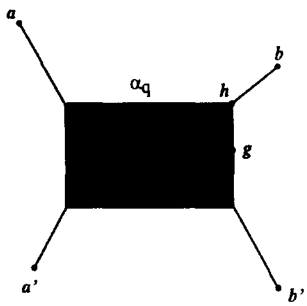


Fig. 1. $T(\{a,a',b,b'\},d')$.

represent the tight span of a (generic) four-point metric space $(\{a,a',b,b'\},d')$ as in Fig. 1. The distance between any two points in this representation can be calculated by following geodesic paths, which, when traversing the shaded rectangle run parallel with one of the sides of the rectangle (i.e. distances in $T(d)$ can be found using the so called 'city block metric' on the rectangle). For the example the distance between a and h is $h(a)$. The quantity α_q defined in Eq. (2) is equal the length of the side labelled by α_q in Fig. 1.

Proposition 4.2. *Suppose that (X,d) is a metric space, and that $S \in \mathcal{S}(X)$. Then $\alpha^d_{\delta_S}$ is equal to α^d_S.*

Proof. In this proof we set $S=\{A,B\}$. First we show that $\alpha^d_S \geqslant \alpha^d_{\delta_S}$. Note that $T(\delta_S)$ is isometric to a closed interval of length one [5], the vertices of which correspond to the functions, $f_A, f_B : X \to \mathbb{R}$, defined by

$$f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f_B(x) := \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Hence, if $f \in T(d)$, then, looking at the definition of the coherency index, we see that the quantity

$$\max_{f' \in VT(\delta_S)} \left\{ \min_{\{x,y\} \notin EK(f')} \{m(f,f')\} \right\}$$

is equal to

$$\tfrac{1}{2} \max \left\{ \min_{a,a' \in A} (f(a) + f(a') - d(a,a')), \min_{b,b' \in B} (f(b) + f(b') - d(b,b')) \right\}.$$

Let $q := \{a,a',b,b'\}$ be a quartet such that $a,a' \in A$, $b,b' \in B$, and for which the isolation index α^d_S is realized, so that α^d_S is equal to the quantity α_q defined in Eq. (2), and let

$h \in VT(d|_{q\times q})$ be the element pictured in Fig. 1. By Property 3 there exists an element $f \in VT(d)$ such that $f|_q = h$. Hence, by the definition of the coherency index, and by considering geodesic paths in Fig. 1, we immediately see that

$$\alpha^d_{\delta_S} \leqslant \frac{1}{2}\max\left\{\min_{a,a'\in A}(f(a)+f(a')-d(a,a')), \min_{b,b'\in B}(f(b)+f(b')-d(b,b'))\right\}$$
$$\leqslant \alpha_q = \alpha^d_S.$$

We now show that $\alpha^d_S \leqslant \alpha^d_{\delta_S}$. Suppose that $f \in VT(d)$, and $f' \in VT(\delta_S)$ are such that they realize the coherency index $\alpha^d_{\delta_S}$. Without loss of generality, assume that $f' = f_B$. Let $a, a' \in A$ be such that

$$f(a) + f(a') - d(a,a')$$

is minimal, so that

$$\alpha^d_{\delta_S} = \tfrac{1}{2}(f(a) + f(a') - d(a,a')).$$

Let $q = \{a, a', b, b'\}$ where $b, b' \in B$, and consider $d' := d|_{q\times q}$. By Property 4 there exists an element $g \in T(d')$ such that $g \leqslant f|_q$. By Property 3, this can be extended to an element in $T(d)$, which we also denote by g. Note that

$$f(a) + f(a') = g(a) + g(a'),$$

otherwise $f(a) + f(a')$ would not be minimal. If $f(a) + f(a') - d(a,a') = 0$, then

$$f(b) + f(b') - d(b,b') = 0$$

by the definition of the coherency index. But this implies that g lies on the geodesic between a, a' and the geodesic between b, b' in Fig. 1. But if this were the case, then $\alpha_q = \alpha^d_S = \alpha^d_{\delta_S} = 0$.

Thus, we can assume that $f(a) + f(a') - d(a,a')$ is greater than zero. Without loss of generality, assume that

$$f(b) + f(b') = d(b,b'),$$

otherwise B would form an independent set in $K(f)$, which would contradict Property 2 since f is a vertex. Thus,

$$f(b) + f(b') = g(b) + g(b') = d(b,b').$$

Since $g(b) + g(b') = d(b,b')$, the element g must be on the geodesic path in $T(d')$ between b and b'. Looking once more at geodesic paths in Fig. 1 we see that

$$g(a) + g(a') = 2\alpha_q + d(a,a'),$$

which implies that

$$\alpha_q = \tfrac{1}{2}(f(a) + f(a') - d(a,a')) = \alpha^d_{\delta_S}.$$

But, by the definition of the isolation index we see that $\alpha_q \geqslant \alpha_S^d$ holds, which completes the proof. □

In fact more can be said on the relationship between the split decomposition of a metric and the coherency index. In [7] Dress states that one only has to look at four point configurations in order to calculate the coherency index of d with respect to a totally decomposable metric d'. Clearly this gives a polynomial algorithm for determining the coherency index in this situation.

4.3. The coherency relation

We define a relation $\sim$ on $M(X)$ using the coherency index, which we call the *coherency relation.* Two metrics $d,d' \in M(X)$ are related under $\sim$ if $\alpha_{d'}^d > 0$ and $\alpha_d^{d'} > 0$. We will show that $\sim$ is an equivalence relation and also give some other characterisations of $\sim$.

As a consequence of Theorem 4.1, if d,d' are prime metrics on a finite set X of distinct type, then $\alpha_{d'}^d = 0$. The following proposition generalizes this fact.

Proposition 4.3. *Let d,d' be two metrics on a finite set X. Then the following statements are equivalent.*
(i) *$\alpha_d^{d'}$ is greater than zero, and*
(ii) *$\{d'' \mid d''$ prime and $\alpha_{d''}^d > 0\} \subseteq \{d'' \mid d''$prime and $\alpha_{d''}^{d'} > 0\}$.*

Proof. (i)⇒(ii): Let $\{d_1,\dots,d_n\}$ be a complete coherent decomposition of d. Then $d = \alpha_1 d_1 + \cdots + \alpha_n d_n$ for some $\alpha_i > 0$. If $\beta := \alpha_d^{d'} > 0$, then $d' = \beta d + d'' = \beta(\alpha_1 d_1 + \cdots + \alpha_n d_n) + d''$ is a coherent decomposition of d'. Hence $\alpha_{d_i}^d$ is greater than zero.
(ii)⇒(i): This follows directly from Corollary 3.5. □

Proposition 4.4. *Let d,d' be two metrics on a set X. The following statements hold:*
(i) *$\sim$ is an equivalence relation on $M(X)$,*
(ii) *$d \sim d'$ if and only if*

$$\{d'' \mid d'' \text{ prime and } \alpha_{d''}^d > 0\} = \{d'' \mid d'' \text{ prime and } \alpha_{d''}^{d'} > 0\}, \tag{3}$$

and
(iii) *$d \sim d'$ if and only if*

$$\{K(f) \mid f \in VT(d)\} = \{K(f') \mid f' \in VT(d')\}. \tag{4}$$

Proof. (ii): This follows immediately from Proposition 4.3.
(i) Clearly the relation $\sim$ is reflexive and symmetric. Hence, it only remains to prove that it is transitive. But this follows easily from (ii).
(iii) It is clear from the definition of the coherency index that if Equality (4) holds, then $d \sim d'$. Thus it suffices to show that $d \sim d'$ implies that Equality (4) holds.

To see why this is true note that, since $\alpha_{d'}^{d} > 0$, for all $f \in VT(d)$ there exists an $f' \in VT(d')$ such that $K(f) \subseteq K(f')$. Also, since $\alpha_{d}^{d'} > 0$, there exists an element $g \in VT(d)$ such that $K(f') \subseteq K(g)$. Hence, we see that $K(f) \subseteq K(g)$, which implies that $K(f) = K(g)$, because f is a vertex of $T(d)$. □

Lemma 4.5. *Let $D = \{d_1, \ldots, d_n\}$ be a coherent decomposition. If two metrics d and d' are both of generic type D, then $d \sim d'$.*

Proof. It is easily seen that $d = \alpha d' + \sum_{i=1}^{n} \alpha_i d_i$ is a coherent decomposition of d for some $\alpha, \alpha_i > 0$. Hence $\alpha_{d'}^{d} > 0$. By symmetry we also have $\alpha_{d}^{d'} > 0$. Hence $d \sim d'$. □

Corollary 4.6. *Let $D = \{d_1, \ldots, d_n\}$ and $D' = \{d'_1, \ldots, d'_n\}$ be two complete coherent decompositions. Then D and D' are of the same type if and only if*

$$d := d_1 + \cdots + d_n \sim d' := d'_1 + \cdots + d'_n.$$

Proof. This is an immediate consequence of Proposition 4.4(iii) and Lemma 4.5. □

4.4. Maximal coherent decompositions and the tight span

In this section we consider $T(X, d)$ as a (non-convex) polytope. The main object of this section is to give a condition for when the polytopal structures of $T(X, d)$ and $T(X, d')$ coincide when d and d' are both sums of maximal coherent decompositions on X. We do this in Corollary 4.12. By the face-poset of a polytope we mean the poset obtained by ordering the faces of the polytope by inclusion.

Lemma 4.7. *Let d be a metric on X. Then the set $\{K(f) \mid f \in VT(d)\}$ determines the set $\{K(f) \mid f \in T(d)\}$, and hence also the face-poset of $T(d)$.*

Proof. Let $f, g \in T(d)$ and define $h := (f + g)/2$. Then $K(h)$ contains only the edges $\{x, y\}$ which are edges of both $K(f)$ and $K(g)$. Since $T(d)$ is compact, for all faces F of $T(d)$ there are two vertices f, g such that the minimal face of $T(d)$ containing f and g is F. But then h lies inside F, which completes the proof. □

Proposition 4.8. *Let d and d' be two metrics on X. Then $d \sim d'$ if and only if*

$$\{K(f) \mid f \in T(d)\} = \{K(f') \mid f' \in T(d')\}. \tag{5}$$

Proof. This proposition follows immediately from Proposition 4.4(iii) and Lemma 4.7. □

Let d and d' be two metrics on X. We say that $d \approx d'$ if there exists a metric d'', of the same type as d such that $d'' \sim d'$. It is easily seen that $\approx$ is an equivalence relation. As we have seen in Property 1, there exists a canonical isometric embedding

$t_d : (X,d) \to (T(d), d_\infty)$. In fact it is clear from Property 2 that $t_d(X)$ is a subset of $VT(d)$.

Proposition 4.9. *Let d be a metric on a finite set X. Then the face-poset of $T(d)$ together with the map t_d determines the set $\{K(f) \mid f \in VT(d)\}$.*

Proof. We define the graph $G(d)$ whose vertices correspond to the faces of $T(d)$ and whose edges are those pairs $\{F_1, F_2\}$ of distinct faces of $T(d)$ such that $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$. Now let $f \in T(d)$, and F be the minimal face of $T(d)$ (with respect to inclusion of faces) containing f. Then $K(f)$ contains the edge $\{x, y\}$, where $x, y \in X$, if and only if F lies on a shortest path of $G(d)$ between $t_d(x)$ and $t_d(y)$, which completes the proof. □

Theorem 4.10. *Let d and d' be two metrics on a finite set X. Then $d \approx d'$ if and only if there exists a face-poset isomorphism*

$$\phi : T(d) \to T(d')$$

such that $\phi \circ t_d(x) = t_{d'}(x)$ for all $x \in X$ and $\phi(VT(d)) = VT(d')$.

Proof. $\Rightarrow$: This follows directly from Proposition 4.8.
$\Leftarrow$: This follows directly from Proposition 4.9. □

In the next lemma we show that every maximal coherent decomposition on X must contain a metric of the form $\alpha \cdot \delta_x$ for each $x \in X$, where δ_x is the split metric associated to the split $S := \{\{x\}, X \setminus \{x\}\}$, and $\alpha > 0$.

Lemma 4.11. *Let $D := \{d_1, \ldots, d_n\}$ be a maximal coherent set on X, and fix $x \in X$. Then there exists $1 \leqslant i \leqslant n$ and $\alpha > 0$ such that δ_x is equal to $\alpha \cdot d_i$.*

Proof. Define $d := d_1 + \cdots + d_n + \delta_x$. It is easy to see that $d(y,z) < d(x,y) + d(x,z)$. Hence $\alpha^d_{\delta_x} > 0$, and so by the maximality of D, it follows that there exist an $1 \leqslant i \leqslant n$ and $\alpha > 0$ such that $\delta_x = \alpha \cdot d_i$. □

Corollary 4.12. *Let $D = \{d_1, \ldots, d_n\}$, and $D' = \{d'_1, \ldots, d'_n\}$ be two maximal coherent sets on a finite set X, and define $d := d_1 + \cdots + d_n$ and $d := d'_1 + \cdots + d'_n$. Then $d \approx d'$ if and only if there is a face-poset isomorphism ϕ of the polytopes $T(d)$ and $T(d')$ (which maps vertices to vertices). Moreover, there exists a (poset) automorphism τ of the face-poset of $T(d)$ such that $\phi \circ t_d \circ \tau = t_{d'}$.*

Proof. First we characterize the set $t_d(X)$ as a subset of the vertices of $T(d)$: the elements of $t_d(X)$ are those elements in $VT(d)$ which have degree one in the graph $G(d)$ (the graph that we defined in the proof of Proposition 4.9). This follows from

Lemma 4.11, as we see that for a fixed x, $K(h_x)$ has only edges $\{x,y\}$, for all $y \in X$. Since the graph $K(f)$ does not have isolated vertices for $f \in T(d)$, we see that there is only one face which is adjacent to $t_d(x)$. The other faces have degree at least two in $G(d)$, since every face of $T(d)$ lies on a geodesic between some pair of elements of $t_d(X)$.

The set $\phi \circ t_d(X)$ is equal to the set $t_{d'}(X)$, as is seen from the fact that ϕ is a poset isomorphism. Hence, there exists a permutation τ of X such that $\phi(t_d(x)) = t_{d'}(\tau(x))$. Define the metric d'' by $d''(x,y) := d'(\tau(x), \tau(y))$. Then d'' and d' have exactly the same face-poset and $d'' \approx d$. Moreover, τ induces an automorphism of this poset by Proposition 4.9. □

4.5. A finiteness theorem

We now use the coherency index to prove that there exist finitely many types of prime metrics on a finite set X, from which it follows immediately that there also exist only finitely many types of coherent decompositions on X.

Theorem 4.13. *Suppose that X is a finite set. Then there exist finitely many types of prime metrics on X.*

Proof. Suppose that d, d' are two distinct metrics on X, with

$$\{K(f)\}_{f \in T(d)} = \{K(f')\}_{f' \in T(d')}.$$

Then, by the definition of coherency index, $\alpha_{d'}^{d} > 0$ and $\alpha_{d}^{d'} > 0$. Hence neither d nor d' is prime. But this implies the result since

$$\bigcup_{f \in T(d)}^{\text{for } d \text{ prime}} \{K(f)\}$$

is a subset of the set of graphs on X, which is finite. □

Corollary 4.14. *If X is a finite set, then there exist finitely many types of maximal coherent sets on X.*

Corollary 4.15. *Each metric d on a finite set X has a complete coherent decomposition.*

Proof. Let $D_1, \ldots, D_N$ be the set of distinct types of coherent decompositions of d of into primes. The corollary now follows from the fact that $d = (1/N)(Nd) = (1/N)d + \cdots + (1/N)d$ is a coherent decomposition of d. □

Note that Theorem 4.13 recovers the result [1, Proposition 1.1] of Avis which states that there exist only finitely many extremal metrics on a finite set.

5. Closing remarks

Recall that $M(X)$ is an $\binom{n}{2}$-dimensional convex cone, the facets of which arise from triangle inequalities for triples of elements in X (see [1, Proposition 1.1]). The cone $M(X)$ is generated by a finite set of extreme rays. It can be easily seen that an extremal metric is also a prime metric. However, the converse of this statement is not true. For example, the 1-skeleton of a cube is prime [3, Proposition 4] but not extremal.

Let $M(d)$ denote the set of all coherent components of d. In [3, p. 97] it is pointed out that $M(d)$ is a closed convex subcone of $M(X)$, so that

$$M(X) = \bigcup_{d \in M(X)} M(d)$$

constitutes an interesting finite stratification of $M(X)$. Originally it was hoped that this stratification would involve only simplicial cones, however, as we have already stated in the introduction, this is not true as a consequence of the result contained in [13]. In a follow-up paper we will determine the cones $M(d)$ for six-point metrics.

According to [1] there are at least 2^{cn^2} extremals on a finite set X of cardinality n, where $c > 0$ is a constant. Since every extremal is also prime this gives us a lower bound on the number of primes on a finite set. It was also shown in [16] that there are at most $2^{c'n^2}$ extremals on a finite set X of cardinality n, where $c' > 0$ is a constant, and we believe that such an upper bound should also hold for prime metrics.

As stated previously in Section 4.2 there exists a polynomial time algorithm for computing the coherency index of a metric with respect to a totally decomposable metric. We believe that the coherency index of a metric defined on a set X of cardinality m, with respect to another whose only prime components are extensions of metrics on a set Y of fixed cardinality, can be computed using an algorithm that is polynomial in m. However, at this point in time it is not clear to us how hard it is to calculate a complete coherent decomposition of a metric d.

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